Rutgers University: Complex Variables and Advanced Calculus Written Qualifying Exam

January 2006 Day 1: Problem 2 Solution

Exercise. Let $f(z) = z^{10} - 10z^3 + 2\sin z + 1$. Determine how many zeros f can have in the unit disk, and then prove your statement.

Solution.

Rouche's Theorem: f, g analytic in open set U and γ a simple path in U, with its interior contained in U and with parameter interval I. If f has no zeros on $\gamma(I)$ and $|f(z) - g(z)| \leq |g(z)|$ on $\gamma(I)$ then f and g have the same number of zeros, counting order, inside γ

$$\begin{aligned} f(z) &= z^{10} - 10z^3 + 2\sin z + 1 \\ &= g(z) + h(z) \end{aligned}$$

Let $g(z) = 2\sin z + 1$ and $h(z) = z^{10} - 10z^3$

$\forall z \in \delta D$, $|g(z)| \leq 2 + 1 = 3$ and

$$\begin{aligned} |h(z)| &= |z^{10} - 10z^3| \\ &= |z^3||10 - z^7| \\ &= |10 - z^7| \\ &\geq 10 - |z|^7 \\ &= 10 - 1 \\ &= 9 \end{aligned}$$

$\implies |g(z)| < |h(z)|$ on δD

By Rouche's Theorem, f and h have the same number of zeros in D.

$$h(z) = z^{10} - 10z^3 = z^3(z^7 - 10) := 0 \quad \text{if} \quad z^3 = 0 \quad \text{or} \quad z^7 - 10 = 0$$

But if $|z| < 1$, then it must be true that $z^3 = 0$

$\implies z = 0$

So, h has 3 zeroes in the unit disk.

Thus, f has 3 zeros in the unit disk.